THE HOPF BIFURCATION

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1. Introduction

The purpose of this project is to analyse the particularities of a Hopf bifurcation by studying the equilibrium points of the orbit of one of the objects in a 3-body system in two dimensions.

2. Definitions

Definition 2.1. A *bifurcation point* is one where the type of a phase portrait changes, after a variation has been made to the value of the parameter of a dynamical system [1].

Bifurcation points can be divided into local (if they can be detected purely by a stability analysis of the equilibria) or global. In this project we are going to consider a specific type of local bifurcation.

Definition 2.2. A *Hopf bifurcation* is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane. It can be expected that a small amplitude limit cycle will branch from the fixed point [2].

3. The set up

Consider the Sun, the Earth and a satellite in a rotating system of coordinates. After normalization of masses and distances, their positions are $(-\mu, 0)$, $(1-\mu, 0)$ and (x, y) respectively. Also, let the satellite's momentum be $\mathbf{p} = (p_x, p_y)$.

Simplifying the physical system, we know the Sun and the Earth are in constant movement. Due to the gravitational attraction (this is, Newton's second law of motion), the Sun and the Earth exert forces on the satellite. Since these forces are not equal, the satellite also moves. But, in some instances, the sum of the forces applied to the satellite is equal to zero, making the satellite stop in what we call an equilibrium point or fixed point (where the momentum $\mathbf{p}$ vanishes). Except the Sun and the Earth do not stop, so soon the forces exerted on the satellite vary, making it go into orbit once more.

The differential equations that influence a satellite moving in this system (in a rotating system of coordinates) are approximated by this system of

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differential equations:

$$\left.\begin{array}{ll} \dot{x} = p_x, & \dot{p}_x = 2p_y + x - \frac{\mu_1(x-\mu_2)}{r_2^3} - \frac{\mu_2(x+\mu_1)}{r_1^3}, \\ \dot{y} = p_y, & \dot{p}_y = -2p_x + y - \frac{\mu_1 y}{r_2^3} - \frac{\mu_2 y}{r_1^3}, \end{array}\right\} \tag{3.1}$$

where $\mu_1 = \mu$, $\mu_2 = 1 - \mu$, $r_1^2 = y^2 + (x + \mu_1)^2$ and $r_2^2 = y^2 + (x - \mu_2)^2$.

4. Finding critical points

(3.1) has five equilibrium points. Three of these lie on the x-axis, but the ones we are interested in have position $L_4 = (-\mu + \frac{1}{2}, \frac{1}{2}\sqrt{3})$ and $L_5 = (-\mu + \frac{1}{2}, -\frac{1}{2}\sqrt{3})$ attained by solving (3.1) when $\mathbf{p} = (0, 0)$. Since the second one is a reflection about the x-axis of the first one, we will only consider L_4 and expect L_5 to behave in the same way.

To find L_4's linear stability I computed the Jacobian matrix of the system (3.1) (as in [3]) on Maple and substituted L_4 into it, then obtaining the following pair of complex conjugate eigenvalues:

$$\begin{bmatrix} \frac{1}{2}\sqrt{-2 + \sqrt{27\mu^2 - 27\mu + 1}} \\ \frac{1}{2}\sqrt{-2 - \sqrt{27\mu^2 - 27\mu + 1}} \end{bmatrix}. \tag{4.1}$$

The linear type of L_4 will change when these two eigenvalues are equal. This happens for two values of μ:

$$\mu_{c1} = \frac{1}{2} - \frac{1}{18}\sqrt{69} \text{ and } \mu_{c2} = \frac{1}{2} + \frac{1}{18}\sqrt{69},$$

which can be found by plotting the real and the imaginary parts of the eigenvalues of the linearized differential equations at L_4, or by simply equating the second radicand to zero and solving for μ.

5. Phase portraits at either side of critical points

In figure 1 there are some plots of solutions of (3.1) at the left of μ_{c1}. We can see how the amplitude of the limit cycle in these ellipses decreases as it reaches μ_{c1}. Plotting the phase portrait with three representative solution curves would show a much clearer limit cycle. When μ crosses μ_{c1}, the phase portraits stop having a limit cycle and evolve into simple spirals (observe figure 2).

μ_{c1} and μ_{c2} are symmetric about $\mu = \frac{1}{2}$ so, if a limit cycle branches towards the left of μ_{c1}, another one will branch towards the right of μ_{c2}. The space between these two critical μs contains spirals and L_4 is hyperbolic since the eigenvalues are complex (with real part not equal to zero). Between $\mu = 0$ and μ_{c1}, and between μ_{c2} and $\mu = 1$, L_4 is elliptic with eigenvalues that are purely imaginary.

6. Conclusion

In conclusion, there is in fact a Hopf bifurcation in our 3-body system in two dimensions since L_4 loses stability as the eigenvalues in (4.1) cross the imaginary axis of the complex plane, and we have been able to see (in §5) how a limit cycle branches from the fixed point.

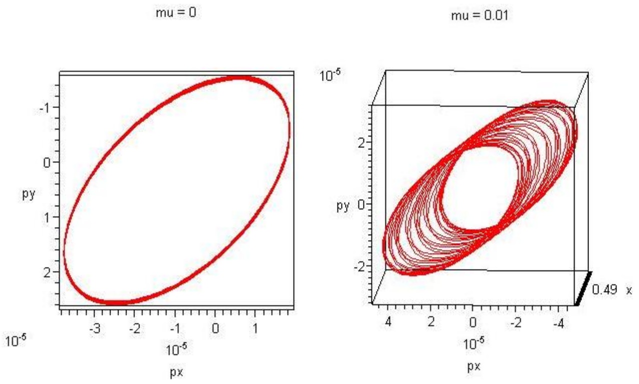

FIGURE 1. $0 \leq \mu < \mu_{c1}$

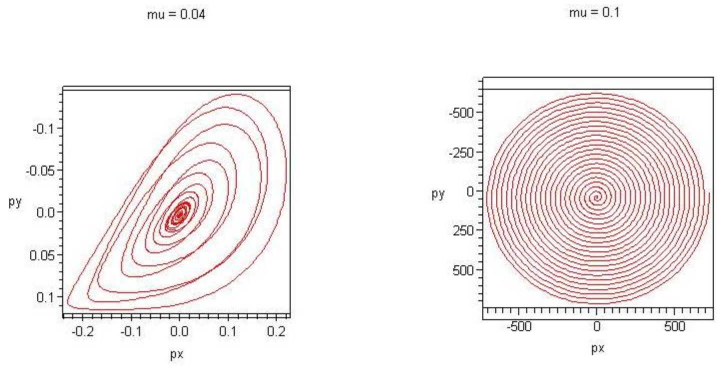

FIGURE 2. $\mu_{c1} \leq \mu \leq \frac{1}{2}$

REFERENCES

[1] Arrowsmith, D. K., & Place, C. M. 1992, *Dynamical Systems: Differential equations, maps and chaotic behaviour*, Chapman & Hall

[2] Wikipedia, *Hopf bifurcation*, available from http://en.wikipedia.org/wiki/Hopf_bifurcation

[3] Iooss, Gérard, & Joseph, Daniel D. 1980, *Elementary Stability and Bifurcation Theory*, Springer-Verlag