FOURIER SERIES AND THE GIBBS PHENOMENON

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1. Introduction

My project consisted on finding out about the Gibbs Phenomenon and the relationship between the smoothness of a graph and the rate of convergence of the Fourier Series.

2. What is a Fourier Series?

A Fourier Series is an approximation of a function by a trigonometric polynomial [1]. This is given by:

$$f(x) \approx a_0/2 + \sum_{k=1}^{n}(a_k cos(kx) + b_k sin(kx)) \tag{2.1}$$

where the Fourier coefficients are given by:

$$a_0 = 1/\Pi \int_{-\Pi}^{\Pi} f(x) \tag{2.2}$$

$$a_k = 1/\Pi \int_{-\Pi}^{\Pi} f(x) cos(kx) \tag{2.3}$$

$$b_k = 1/\Pi \int_{-\Pi}^{\Pi} f(x) sin(kx) \tag{2.4}$$

The basic idea of a Fourier approximation is that as k increases, the function approaches the original one more and more. At some point these should in fact be equal.

Figure 1 shows an example of this on a continuous (everywhere differentiable) graph.

The peculiar manner in which the Fourier Series of a piecewise continuously differentiable periodic function behaves at a discontinuity is called the Gibbs Phenomenon [2]. We will see an example of this in §4.

3. Historical background of the Gibbs Phenomenon

In 1898 Albert Michelson (a Polish physicist) developed a machine that could calculate and re-unify the Fourier Series [3]. When the Fourier coefficients for a square wave were introduced, the graph oscillated at the discontinuities. This still happened even when the number of Fourier coefficients was increased. Michelson thought the machine was faulty, but the American physicist J. Willard Gibbs said, in 1899, that these oscillations were a mathematical phenomenon and would always occur when applying Fourier Series to discontinuous functions.

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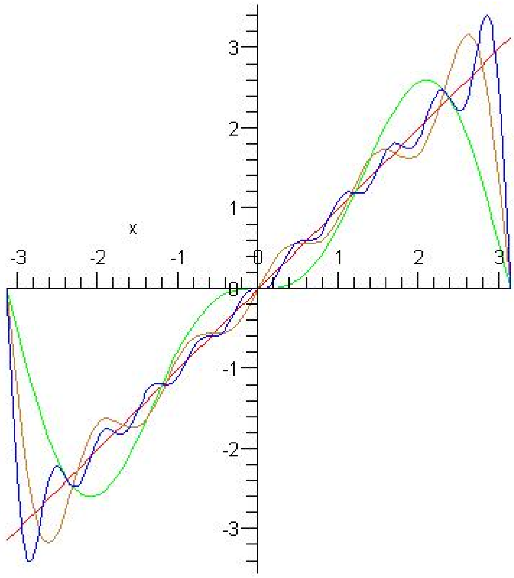

FIGURE 1. $y = |x|$

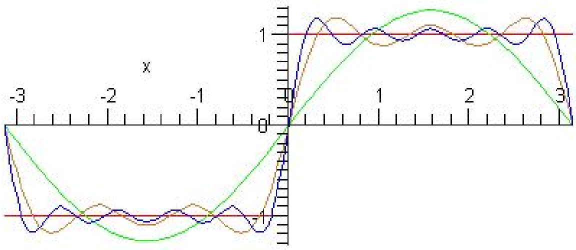

FIGURE 2. $y = -1$ if $-\Pi \leqslant x \leqslant 0$ and $y = 1$ if $0 \leqslant x \leqslant \Pi$

Later on, in 1906, the American mathematician Maxime Bôcher analysed this mathematically and gave it the name of Gibbs Phenomenon.

It must be said, though, that Henry Wilbraham (who was an English mathematician) had discovered this phenomenon and published a paper on it nearly fifty years before, but not many people were aware of it.

4. An example of the Gibbs Phenomenon

Figure 2 shows an example of the Gibbs Phenomenon.

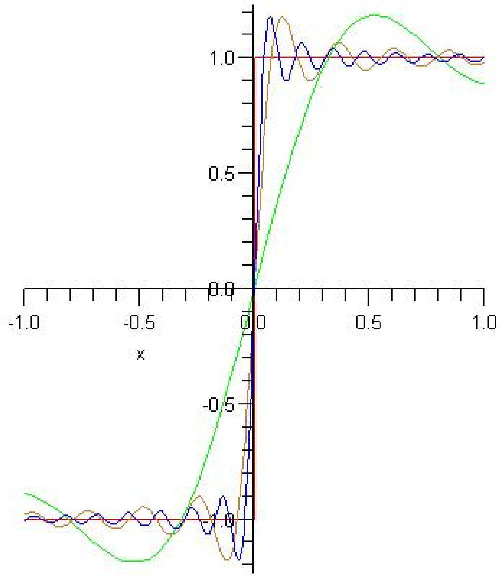

FIGURE 3. $y = -1$ if $-\Pi \leqslant x \leqslant 0$ and $y = 1$ if $0 \leqslant x \leqslant \Pi$

We can observe how, as the number of Fourier coefficients increases, the graph tends to look more like the original one in the continuous area ($-\Pi < x < 0$ and $0 < x < \Pi$), but near the discontinuities (around the origin) the amplitude of the wiggles remains constant (see Figure 3 for a closer look).

This suggests that at points of discontinuity the partial sums do not converge smoothly to the mean value. Instead, they miss the target from above and below as if they cannot adapt to the acute turn required at this point.

5. Conclusion

It can then be stated that the smoother a function is, the faster its Fourier Series converges.

References

[1] Adams, Robert A. 2003, *Calculus. A complete course*, Addison Wesley Longman
[2] Wikipedia, *Gibbs Phenomenon*, available from http://en.wikipedia.org/wiki/Gibbs_phenomenon
[3] Richards, D. 2002, *Advanced mathematical methods with Maple*, Cambridge University Press