

# Elliptic Functions

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# Abstract

Elliptic functions are doubly-periodic functions in the complex plane, with a range of interesting properties and applications. I provide definitions and basic theorems that may serve as starting point for further developments in the field. In particular, Weierstrass functions exhibit a set of properties, and satisfy differential equations and addition theorems that are fully proved in these notes.

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# Chapter 1

## Introduction

Elliptic functions are double-periodic functions in the complex plane. The combination of the analytic and algebraic-arithmetic theory of elliptic functions has been at the centre of mathematics since the early part of the nineteenth century [1]. Over the years, these functions have been studied by many mathematicians such as Jacobi, Abel and Weierstrass to name a few. The aim of this project is to study some of the properties and theorems regarding elliptic functions.

As a starting point, we will look at some definitions that will be used in different sections of the project.

### 1.1 Definitions

**Definition 1.** A function  $f(z)$  is *analytic* in the region  $R$  if its derivative  $f'(z)$  exists for all  $z \in R$ .

Further,  $f(z)$  is analytic in  $z = z_0$  if there exists a neighbourhood  $|z - z_0| < \delta$  such that  $f'(z)$  exists for all  $z$  inside that neighbourhood [2].

The concept of periodicity is expressed mathematically by saying that, for the periodic function  $f(z)$ , there exists a complex number  $a$  such that

$$f(z + a) = f(z)$$

for all values of  $z$  for which  $f(z)$  is analytic [3]. So after  $z$  goes from  $z_0$  to  $z_0 + a$ , the function starts repeating its behaviour over future periods (from  $z_0 + a$  to

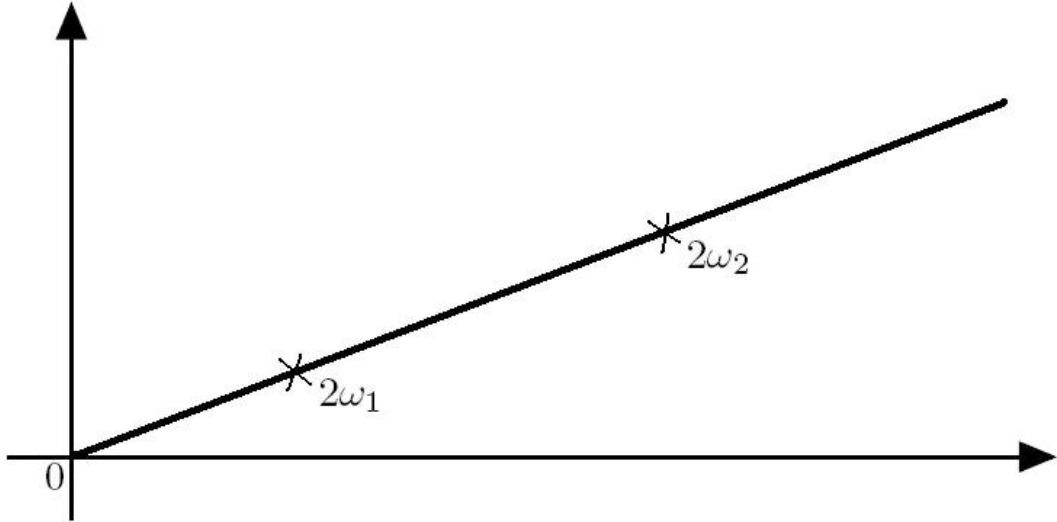


Figure 1.1: Case when  $\frac{\omega_2}{\omega_1} \in \mathbb{R}$ .

$z_0 + 2a$ , from  $z_0 + 2a$  to  $z_0 + 3a$  and so on).

Here, we will need the concept of double-periodicity.

**Definition 2.** A function  $f(z)$  is *doubly-periodic* with periods  $2\omega_1$  and  $2\omega_2$  if, for all  $z$  for which  $f(z)$  is analytic,

$$f(z + 2\omega_1) = f(z) \quad \text{and} \quad f(z + 2\omega_2) = f(z).$$

In §2.1 we will discuss the possibility of covering the whole of the  $z$ -plane with period-parallelograms. If the ratio  $\frac{\omega_2}{\omega_1}$  is in the real numbers  $\mathbb{R}$ , the periods  $2\omega_1$  and  $2\omega_2$  would be in the same direction, as in Figure 1.1, and the parallelograms would collapse into a single line. Also, if  $\frac{\omega_2}{\omega_1} \in \mathbb{Q}$ , then the function  $f(z)$  becomes simply-periodic; and if the ratio  $\frac{\omega_2}{\omega_1}$  is irrational, then the function becomes a constant.

For the purpose of this project we can then assume that the ratio  $\frac{\omega_2}{\omega_1}$  is not real.

**Definition 3.** A *meromorphic function* is one which is analytic at all points of the finite plane, except in a set of poles. [2].

# Chapter 2

## Elliptic functions

**Definition 4.** An *elliptic function* is a doubly-periodic function which is analytic at all points except at poles, its only singularities in the finite part of the plane [4].

Alternatively, an elliptic function can be defined as a meromorphic function in the complex plane with the two periods  $2\omega_1$  and  $2\omega_2$ . As we said in §1.1, we can assume that  $\text{Im}(\frac{\omega_2}{\omega_1}) > 0$ , so the rotation from  $2\omega_1$  to  $2\omega_2$  on the  $z$ -plane is performed anticlockwise.

Note that throughout this project, unless stated otherwise, elliptic functions will have periods  $2\omega_1$  and  $2\omega_2$ , following the notation of Whittaker and Watson in their book *A course of modern analysis* [4].

### 2.1 Period-parallelograms

The whole of the  $z$ -plane may be covered throughout with parallelograms called *period-parallelograms* or *meshes*.

A period-parallelogram that does not hold the points  $\omega$  (the periods of the function), such that  $f(z + \omega) = f(z)$ , in its inside or boundary (except vertices), is called a fundamental period-parallelogram. In Figure 2.1,  $m, n \in \mathbb{Z}$  and the fundamental period-parallelogram has been coloured in with yellow.

Obviously, there can be more than one fundamental period-parallelogram for each elliptic function, and the image of these under parallel translations are also

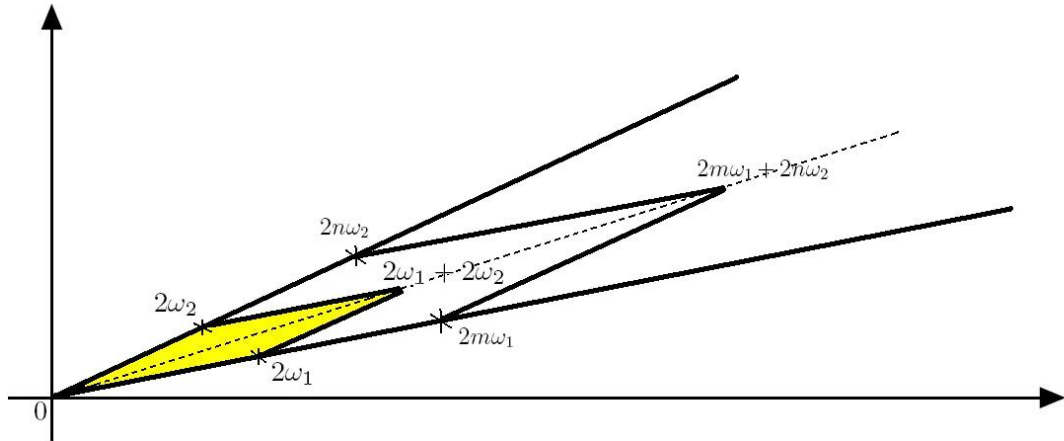


Figure 2.1: Period-parallelograms in the  $z$ -plane.

referred to as fundamental period-parallelograms.

For all  $z$ , the points  $z, z + 2\omega_1, \dots, z + 2m\omega_1 + 2n\omega_2, \dots$  occupy corresponding positions in meshes. Any pair of such points are congruent to one another. So, for instance, take the points  $z$  and  $z'$ , then

$$z' = z \pmod{2\omega_1, 2\omega_2}.$$

An elliptic function assumes the same value at every set of congruent points. So, its values in any mesh are a repetition of its values in any other. Thus, the study of any elliptic function can be reduced to consideration of its behaviour in a fundamental period-parallelogram.

To finalise this chapter, it is worth mentioning that the derivative of an elliptic function is an elliptic function too. Also, the addition, subtraction, multiplication and division of elliptic functions give elliptic functions as well.

# Chapter 3

## General properties of elliptic functions

Recall the concept of a mesh in §2.1. In order to integrate this area, we must translate (without rotation) such a mesh. This is then called a cell (and it has the same values as any mesh). Figure 3.1 shows a cell with vertices on the points  $t$ ,  $t+2\omega_1$ ,  $t+2\omega_1+2\omega_2$  and  $t+2\omega_2$ . The contour of the parallelogram constructed will be denoted by  $C$ . This contour can be divided into four sections, starting in the point  $t$  and reaching every vertex in the anticlockwise direction. These will be denoted by  $\gamma_1$ ,  $\gamma_2$ ,  $\gamma_3$  and  $\gamma_4$ .

We are going to study some general properties of elliptic functions. Many textbooks refer to some of these properties as *Liouville's Theorems* (for instance, Akhiezer's *Elements of the theory of elliptic functions* [5]) for their relation with the well-known Liouville's Theorem:

**Theorem 5.** *Let  $f(z)$  be analytic for all  $z$  and let  $|f(z)| < K$  for all  $z$ , where  $K$  is a constant (so that  $|f(z)|$  is bounded as  $|z| \rightarrow \infty$ ). Then  $f(z)$  is constant.*

The main reference used for the study of the following theorems has been [4]. However, [5] and [6] have been invaluable for the understanding of some of the proofs.

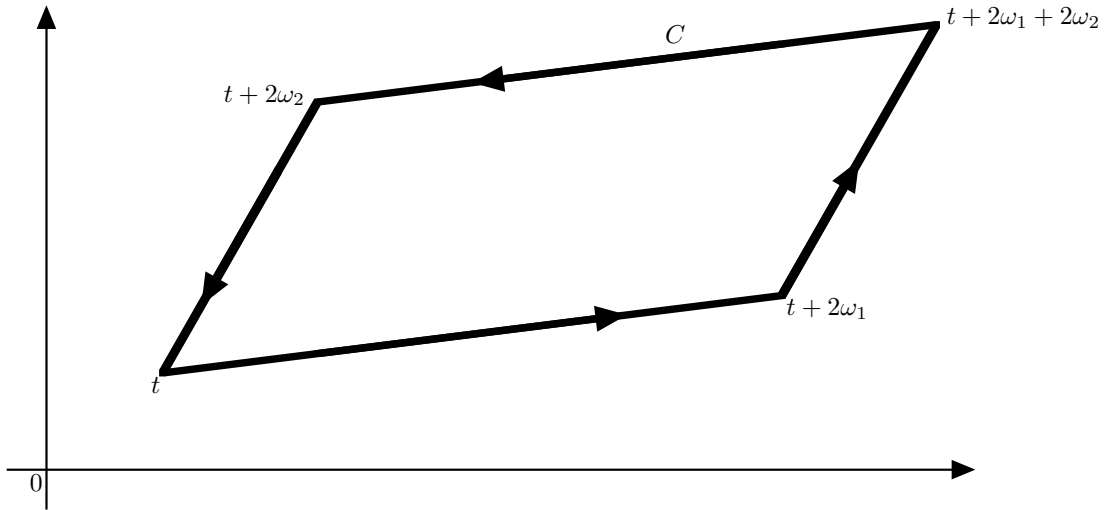


Figure 3.1: Elliptic function cell.

### 3.1 Liouville's theorems

**Proposition 6.** *The number of poles of an elliptic function  $f(z)$  in any cell, counting multiplicity, is finite.*

We shall prove this property by contradiction.

*Proof.* Let us assume that the number of poles in a cell is infinite.

If we choose a sequence of such poles, these would contain a convergent subsequence. This limit point is an essential singularity of the elliptic function. But, from Definition 4, we know that the only finite singularities an elliptic function can have are poles, so this is a contradiction.  $\square$

**Proposition 7.** *The number of zeros of an elliptic function  $f(z)$  in any cell, counting multiplicity, is finite.*

Note that if the number of zeros in a cell is infinite, then the reciprocal of the function has an infinite number of poles in this cell.

The proof of Proposition 7 follows directly from the proof of Proposition 6.

**Proposition 8.** *The sum of the residues of an elliptic function  $f(z)$  at its poles in any cell is zero.*

*Proof.* Consider any cell in the plane (see Figure 3.1 and the beginning of this chapter for notation on vertices and contour sections).

The sum of the residues of  $f(z)$  at its poles inside the cell  $C$  is:

$$\frac{1}{2\pi i} \int_C f(z) dz = \frac{1}{2\pi i} \left\{ \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{\gamma_3} f(z) dz + \int_{\gamma_4} f(z) dz \right\}. \quad (3.1)$$

Writing  $z + 2\omega_1$  and  $z + 2\omega_2$  for the second and third integrals respectively, and from the periodic properties of  $f(z)$ , we have:

$$\int_{\gamma_2} f(z) dz = - \int_{\gamma_4} f(z + 2\omega_1) dz = - \int_{\gamma_4} f(z) dz \quad (3.2)$$

and

$$\int_{\gamma_3} f(z) dz = - \int_{\gamma_1} f(z + 2\omega_2) dz = - \int_{\gamma_1} f(z) dz. \quad (3.3)$$

So, the right-hand side of equation (3.1) becomes

$$\frac{1}{2\pi i} \left\{ \int_{\gamma_1} f(z) dz - \int_{\gamma_4} f(z) dz - \int_{\gamma_1} f(z) dz + \int_{\gamma_4} f(z) dz \right\}. \quad (3.4)$$

Therefore,

$$\frac{1}{2\pi i} \int_C f(z) dz = 0. \quad (3.5)$$

□

**Proposition 9.** *An elliptic function  $f(z)$  with no poles in a cell is merely a constant.*

*Proof.* From the definition of analyticity (Definition 1), we know that if  $f(z)$  has no poles in (and on the boundary of) a cell, then  $f(z)$  is analytic (and therefore bounded) in that cell. This means that there exists a number  $K$  such that  $|f(z)| < K$  when  $z$  is in the cell. From the periodic properties of  $f(z)$ , the function  $f(z)$  is analytic throughout, and  $|f(z)| < K$  for all  $z$ . Therefore, by Theorem 5,  $f(z)$  is a constant. □

**Proposition 10.** *The number of roots of the equation  $f(z) = c$  which lie in any cell depends only on the elliptic function  $f(z)$ , and not on the constant  $c$ .*

*Proof.* Consider the following equation:

$$f(z) - c = 0. \quad (3.6)$$

The difference between the number of zeros and the number of poles, which lie in the cell  $C$ , of equation (3.6) is

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - c} dz. \quad (3.7)$$

Since  $2\omega_1$  and  $2\omega_2$  are the periods of the elliptic function  $f(z)$ ,

$$f'(z + 2\omega_1) = f'(z + 2\omega_2) = f'(z). \quad (3.8)$$

Hence, the integrand in (3.7) is again an elliptic function with the periods  $2\omega_1$  and  $2\omega_2$ .

It follows from the proof of Proposition 8, that the integral (3.7) is equal to zero. Since the difference between the number of zeros and the number of poles of equation (3.6) is zero, the number of zeros is in fact equal to the number of poles. But any pole of equation (3.6) is a pole of  $f(z)$  and so the number of zeros of  $f(z) - c$  is equal to the number of poles of  $f(z)$ , which is independent of the constant  $c$ .  $\square$

This number is called the *order* of the elliptic function and is equal to the number of poles of  $f(z)$  in the cell.

**Proposition 11.** *The sum of the affixes of a set of irreducible zeros of an elliptic function is congruent to the sum of the affixes of a set of irreducible poles.*

To prove this proposition we must show that:

$$\Sigma \text{ affixes of zeros} - \Sigma \text{ affixes of poles} = \text{period.}$$

*Proof.* With the division of contour  $C$  as in the proof of Proposition 8, we know

that the difference between the sums mentioned above is:

$$\begin{aligned}
\frac{1}{2\pi i} \int_C \frac{zf'(z)}{f(z)} dz &= \frac{1}{2\pi i} \left\{ \int_{\gamma_1} \frac{zf'(z)}{f(z)} dz + \int_{\gamma_2} \frac{zf'(z)}{f(z)} dz + \int_{\gamma_3} \frac{zf'(z)}{f(z)} dz + \right. \\
&\quad \left. + \int_{\gamma_4} \frac{zf'(z)}{f(z)} dz \right\} \\
&= \frac{1}{2\pi i} \left\{ \int_{\gamma_1} \left[ \frac{zf'(z)}{f(z)} - \frac{(z+2\omega_2)f'(z+2\omega_2)}{f(z+2\omega_2)} \right] dz + \right. \\
&\quad \left. + \int_{\gamma_4} \left[ \frac{zf'(z)}{f(z)} - \frac{(z+2\omega_1)f'(z+2\omega_1)}{f(z+2\omega_1)} \right] dz \right\} \\
&= \frac{1}{2\pi i} \left\{ -2\omega_2 \int_{\gamma_1} \frac{f'(z)}{f(z)} dz - 2\omega_1 \int_{\gamma_4} \frac{f'(z)}{f(z)} dz \right\} \\
&= \frac{1}{2\pi i} \{ -2\omega_2 [\ln f(z)]_{\gamma_1} - 2\omega_1 [\ln f(z)]_{\gamma_4} \}
\end{aligned}$$

Recall that the limit points for  $\gamma_1$  are  $t$  to  $t+2\omega_1$  and those for  $\gamma_4$  are  $t+2\omega_2$  to  $t$ . The value of  $f(z)$  at the points  $t+2\omega_1$  and  $t+2\omega_2$  is the same as at point  $t$ , so the values of  $\ln f(z)$  at these points can only differ from the value of  $\ln f(t)$  by integer multiples of  $2\pi i$ . This is, for  $m, n \in \mathbb{Z}$ ,

$$\frac{1}{2\pi i} \int_C \frac{zf'(z)}{f(z)} dz = 2m\omega_1 + 2n\omega_2, \tag{3.9}$$

as required. □

**Proposition 12.** *The order of an elliptic function is (always) bigger than or equal to 2.*

*Proof.* Assume  $f(z)$  is an elliptic function of order 1. This means  $f(z)$  has one simple pole in a cell and, hence, its residue is not equal to zero. This contradicts Proposition 8, so the order of an elliptic function can never be less than 2. □

# Chapter 4

## Weierstrass functions

The simplest (non-trivial) elliptic functions are of order two. Carl G. J. Jacobi (1804-1851) chose a function with two simple poles as the standard function of order two. On the other hand, Karl T. W. Weierstrass (1815-1897) selected one with a double pole [7].

For the study of Jacobian elliptic functions, we can refer to books such as Morse and Feshbach's *Methods of theoretical physics* [3] or Lawden's *Elliptic functions and applications* [8].

Here, we are going to discuss Weierstrassian elliptic functions; in particular, his  $\wp$  function.

### 4.1 The Weierstrass $\wp$ function

The Weierstrass  $\wp$  function is an elliptic function with one second-order pole and two zeros per unit cell [3].

It is easy to think of the simplest function with one second-order pole; this is,  $f(z) = z^{-2}$ . But this, by itself, does not constitute an elliptic function, for it has no periods.

The Weierstrass  $\wp$  function is defined

$$\wp(z) = \frac{1}{z^2} + \sum'_{m,n} \left[ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right], \quad (4.1)$$

where the symbol  $\Sigma'$  means that the summation is made over all combinations

$$|P(z, 1.0, 2.0)|$$

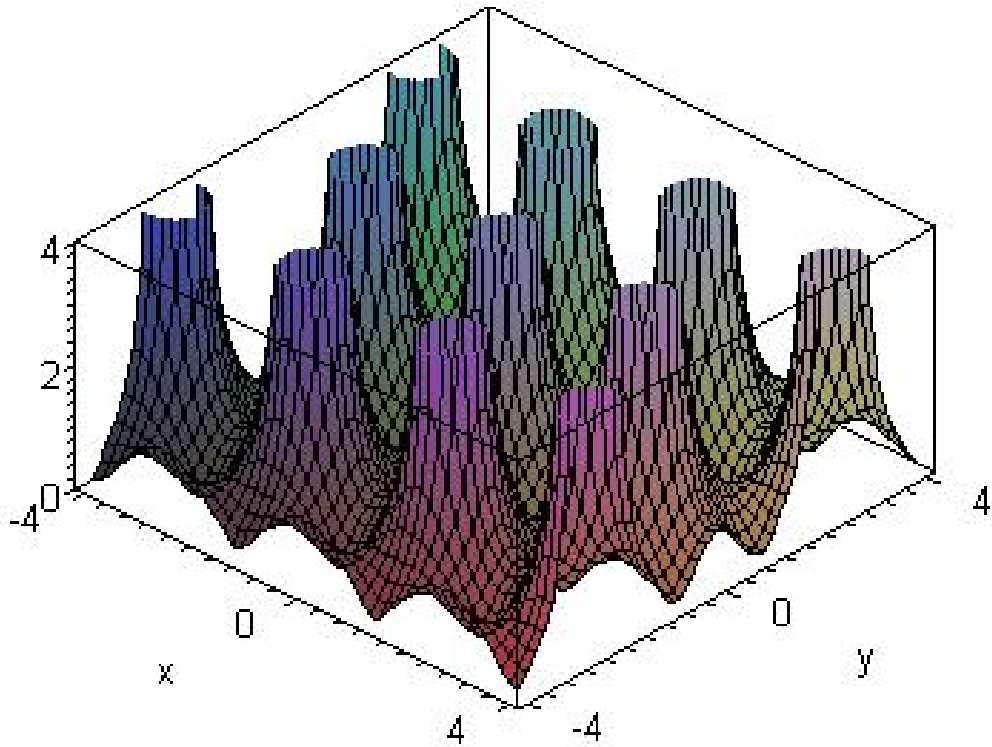


Figure 4.1: Weierstrass  $\wp(z)$  function in absolute value.

of integers  $m$  and  $n$  except for the combination  $m = n = 0$ ;  $2\omega_1$  and  $2\omega_2$  are the periods of the function  $\wp$  [9]. Figure 4.1 shows a particular  $\wp(z)$  function in absolute value in the form  $\wp(z, g_2, g_3)$ .

We will show that  $\wp(z + 2m\omega_1 + 2n\omega_2) = \wp(z)$  in the section that follows.

#### 4.1.1 Properties

**Theorem 13.** *The function  $\wp(z)$  is an elliptic function with the periods  $2\omega_1, 2\omega_2$ . Its poles are given by  $z = 2m\omega_1 + 2n\omega_2$ . It has the further properties:*

- (i) *the principal part of  $\wp(z)$  at  $z = 0$  is  $\frac{1}{z^2}$ ;*

$$(ii) \lim_{z \rightarrow 0} (\wp(z) - \frac{1}{z^2}) = 0;$$

$$(iii) \wp(z) = \wp(-z);$$

$$(iv) \wp'(-z) = -\wp'(z) \text{ [10].}$$

*Proof.* (i) and (ii) follow directly from (4.1).

Now note that

$$\wp(-z) = \frac{1}{z^2} + \sum'_{m,n} \left[ \frac{1}{(z + \Omega_{m,n})^2} - \frac{1}{(\Omega_{m,n})^2} \right], \quad (4.2)$$

where  $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$ .

The set of points  $-\Omega_{m,n}$  is the same as the set  $\Omega_{m,n}$ , which means that the terms of  $\wp(-z)$  are the same as the terms of  $\wp(z)$  but in a different order. But the series for  $\wp(z)$  is absolutely convergent, so the disorder of the terms does not affect the summation of these. Therefore, (iii) holds.

Similarly, consider

$$\wp'(z) = \frac{d}{dz} \wp(z) = -2 \sum'_{m,n} \frac{1}{(z - \Omega_{m,n})^3}. \quad (4.3)$$

Term-by-term differentiation is permitted since the series for  $\wp(z)$  is a uniformly convergent one of analytic functions. Consider at the same time

$$\wp'(-z) = 2 \sum'_{m,n} \frac{1}{(z + \Omega_{m,n})^3}. \quad (4.4)$$

We know that the terms of  $\wp'(-z)$  are the same as of  $-\wp'(z)$  but in a different order because, as before, the sets of points  $-\Omega_{m,n}$  and  $\Omega_{m,n}$  are equal. Also, the sum is not affected either by the derangement of its terms, since the series for  $\wp'(z)$  is absolutely convergent [4]. So, (iv) holds too.

Now observe the following:

$$\wp'(z + 2\omega_1) = -2 \sum'_{m,n} \frac{1}{(z - \Omega_{m,n} + 2\omega_1)^3}. \quad (4.5)$$

The set of points  $\Omega_{m,n} - 2\omega_1$  is the same as  $\Omega_{m,n}$ , so the series for  $\wp'(z + 2\omega_1)$  is a

disorder of the series for  $\wp'(z)$ . Then, from the fact that this series is absolutely convergent,

$$\wp'(z + 2\omega_1) = \wp'(z), \quad (4.6)$$

meaning that  $\wp'(z)$  has period  $2\omega_1$ .

Similarly,  $\wp'(z)$  has period  $2\omega_2$ .

Integrating equation (4.6):

$$\wp(z + 2\omega_1) = \wp(z) + C, \quad (4.7)$$

where  $C$  is a constant of integration. Set  $z = -\omega_1$ , then (4.7) becomes:

$$\wp(-\omega_1 + 2\omega_1) = \wp(\omega_1) = \wp(-\omega_1) + C. \quad (4.8)$$

From (iii),  $\wp(\omega_1) = \wp(-\omega_1)$ , so  $C = 0$ . Therefore,

$$\wp(z + 2\omega_1) = \wp(z),$$

and similarly,

$$\wp(z + 2\omega_2) = \wp(z).$$

So  $\wp(z)$  has periods  $2\omega_1$  and  $2\omega_2$ .

This concludes the proof. □

## 4.2 The differential equation satisfied by $\wp(z)$

**Theorem 14.** *The elliptic function  $\wp(z)$  satisfies the differential equation*

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3, \quad (4.9)$$

where

$$g_2 = 60 \sum'_{m,n} \Omega_{m,n}^{-4}, \quad g_3 = 140 \sum'_{m,n} \Omega_{m,n}^{-6}$$

and  $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$  [10].

The main reference for the proof of this theorem is Whittaker and Watson's *A course of modern analysis* [4].

*Proof.* From the definition of the Weierstrass  $\wp$  function (4.1),

$$\wp(z) - \frac{1}{z^2} = \sum'_{m,n} \left[ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right], \quad (4.10)$$

which is even and analytic in a region of which the origin is an internal point.

Using Taylor's expansions we obtain:

$$\wp(z) - \frac{1}{z^2} = \frac{1}{20}g_2z^2 + \frac{1}{28}g_3z^4 + \text{higher order terms}, \quad (4.11)$$

valid for sufficiently small values of  $|z|$  and where  $g_2$  and  $g_3$  have been set to  $60\sum_{m,n}'\Omega_{m,n}^{-4}$  and  $140\sum_{m,n}'\Omega_{m,n}^{-6}$  respectively.

Rearranging for  $\wp(z)$  equation (4.11) and differentiating,

$$\wp'(z) = -2\frac{1}{z^3} + \frac{1}{10}g_2z + \frac{1}{7}g_3z^3 + \text{higher order terms}. \quad (4.12)$$

Now we can cube equation (4.11) and square equation (4.12) to obtain:

$$\wp^3(z) = \frac{1}{z^6} + \frac{3}{20}g_2\frac{1}{z^2} + \frac{3}{28}g_3 + \text{higher order terms} \quad (4.13)$$

and

$$\wp'^2(z) = 4\frac{1}{z^6} - \frac{2}{5}g_2\frac{1}{z^2} - \frac{4}{7}g_3 + \text{higher order terms} \quad (4.14)$$

respectively.

Next subtract four times equation (4.13) of equation (4.14):

$$\wp'^2(z) - 4\wp^3(z) = -g_2\frac{1}{z^2} - g_3 + \text{higher order terms}. \quad (4.15)$$

If you substitute  $\frac{1}{z^2}$  for  $\wp(z)$  in this last equation, you will get the following function:

$$\wp'^2(z) - 4\wp^3(z) + g_2\wp(z) + g_3 = \text{higher order terms}. \quad (4.16)$$

This function is an elliptic function analytic at the origin and, thus, analytic

at all congruent points. But these points are the only possible singularities, so this elliptic function has no singularities and hence, by Proposition 9, the function is a constant.

Making  $z$  tend to zero, so as to see what will become of the right-hand side of (4.16), shows that this constant is in fact zero.

Therefore  $\wp(z)$  satisfies the differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3,$$

where the invariants  $g_2$  and  $g_3$  are as follows:

$$g_2 = 60 \sum'_{m,n} \Omega_{m,n}^{-4} \quad \text{and} \quad g_3 = 140 \sum'_{m,n} \Omega_{m,n}^{-6}.$$

□

### 4.2.1 Converse proof

Alternatively, one may be given

$$\left(\frac{dy}{dx}\right)^2 = 4y^3 - g_2y - g_3. \quad (4.17)$$

If two numbers  $\omega_1$  and  $\omega_2$  can be determined such that

$$g_2 = 60 \sum'_{m,n} \Omega_{m,n}^{-4} \quad \text{and} \quad g_3 = 140 \sum'_{m,n} \Omega_{m,n}^{-6},$$

then the general solution of the differential equation (4.17) is

$$y = \wp(\pm z + \alpha), \quad (4.18)$$

where  $\alpha$  is the constant of integration.

Since the Weierstrass  $\wp(z)$  function is even,

$$y = \wp(z \pm \alpha), \quad (4.19)$$

and so, without loss of generality, the solution to the equation is

$$y = \wp(z + \alpha). \tag{4.20}$$

# Chapter 5

## The constants $e_1, e_2, e_3$

We are now going to relate the Weierstrass  $\wp$  function to the constants  $e_1, e_2$  and  $e_3$  [4] by showing that:

- $\wp(\omega_1), \wp(\omega_2)$  and  $\wp(\omega_3)$ , where  $\omega_3 = -\omega_1 - \omega_2$ , are all unequal; and
- if  $\wp(\omega_1) = e_1, \wp(\omega_2) = e_2$  and  $\wp(\omega_3) = e_3$  then the constants  $e_1, e_2$  and  $e_3$  are the roots of the elliptic curve  $4t^3 - g_2t - g_3 = 0$ , where  $g_2$  and  $g_3$  are as in §4.2.

### 5.1 $\wp(\omega_1), \wp(\omega_2), \wp(\omega_3)$

First consider  $\wp'(\omega_1)$ . From the oddity of the derivative of the Weierstrass  $\wp$  function, we know

$$\wp'(\omega_1) = -\wp'(-\omega_1).$$

On the other hand, from its periodic properties,

$$-\wp'(-\omega_1) = -\wp'(2\omega_1 - \omega_1) = -\wp'(\omega_1).$$

But  $\wp'(\omega_1) \neq -\wp'(\omega_1)$  since we already established that  $\wp'(z)$  is not even, but odd. Hence,  $\wp'(\omega_1) = 0$ .

Similarly,

$$\wp'(\omega_2) = -\wp'(-\omega_2) = -\wp'(2\omega_2 - \omega_2) = -\wp'(\omega_2),$$

so  $\wp'(\omega_2) = 0$ ; and

$$\begin{aligned}
\wp'(\omega_3) &= \wp'(-\omega_1 - \omega_2) \\
&= -\wp'(\omega_1 + \omega_2) \\
&= -\wp'(\omega_1 + \omega_2 - 2\omega_1 - 2\omega_2) \\
&= -\wp'(-\omega_1 - \omega_2) \\
&= -\wp'(\omega_3),
\end{aligned}$$

so  $\wp'(\omega_3) = 0$ .

Now,

$$\wp'(z) = -2 \sum_{m,n} \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^3}.$$

From this, it is easy to see that  $\wp'(z)$  has poles of order three at the points congruent to the origin (these make the denominator in the summation zero, so the function blows up) as its only singularities. Then, from Proposition 11,  $\wp'(z)$  has (only) three irreducible zeros, and so, the only zeros of  $\wp'(z)$  are the points congruent to  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ .

Now consider the function  $\wp(z) = e_1$ . The function vanishes at  $z = \omega_1$  and, since  $\wp'(\omega_1) = 0$ , it has a double zero at  $z = \omega_1$ . Since  $\wp(z)$  is an elliptic function of order 2, it only has two irreducible poles, which implies that the only zeros of  $\wp(z) = e_1$  are congruent to  $\omega_1$ . Similarly, the only zeros of  $\wp(z) = e_2$  and of  $\wp(z) = e_3$  are double zeros at points congruent to  $\omega_2$  and  $\omega_3$  respectively.

Observe that if  $e_1 = e_2$  then  $\wp(z) = e_1$  has a zero at  $\omega_2$  which is a point not congruent to  $\omega_1$ , so  $e_1 \neq e_2$ . If  $e_1 = e_3$  then  $\wp(z) = e_1$  has a zero at  $\omega_3$  which is again a contradiction, so  $e_1 \neq e_3$ . And finally, if  $e_2 = e_3$  then  $\wp(z) = e_2$  has a zero at  $\omega_3$  which is not a point congruent to  $\omega_2$ , so  $e_2 \neq e_3$ . So  $e_1$ ,  $e_2$  and  $e_3$  are all distinct.

Therefore,  $\wp(\omega_1)$ ,  $\wp(\omega_2)$  and  $\wp(\omega_3)$ , where  $\omega_3 = -\omega_1 - \omega_2$ , are all distinct.

## 5.2 $e_1, e_2, e_3$ and the curve $4t^3 - g_2t - g_3 = 0$

In §4.2 we saw the differential equation satisfied by  $\wp(z)$  (equation (4.9)):

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3.$$

Also, we just discussed that  $\wp'(z)$  vanishes at  $z = \omega_1, \omega_2$  and  $\omega_3$  (§5.1).

From these two remarks, it follows that  $4\wp^3(z) - g_2\wp(z) - g_3$  vanishes when  $\wp(z) = e_1, e_2$  or  $e_3$ . This means that the constants  $e_1, e_2$  and  $e_3$  are the roots of the elliptic curve

$$4t^3 - g_2t - g_3 = 0$$

with  $g_2$  and  $g_3$  as before, and from the formulae connecting the roots of equations with their coefficients, we have:

- (i)  $e_1 + e_2 + e_3 = 0$ ;
- (ii)  $e_2e_3 + e_3e_1 + e_1e_2 = -\frac{1}{4}g_2$ ; and
- (iii)  $e_1e_2e_3 = \frac{1}{4}g_3$ .

# Chapter 6

## Representation of elliptic functions in terms of $\wp$ and $\wp'$

**Theorem 15.** *Any elliptic function  $f(z)$  can be expressed in terms of the Weierstrassian elliptic functions  $\wp(z)$  and  $\wp'(z)$  with the same periods, the expression being rational in  $\wp(z)$  and linear in  $\wp'(z)$  [4].*

This is to say,  $f(z)$  may be written as:

$$f(z) = R_1[\wp(z)] + R_2[\wp(z)]\wp'(z)$$

where  $R_1$  and  $R_2$  are two rational functions.

Next, we shall see how this theorem is obtained.

### 6.1 The proof

First, let  $f(z)$  be any elliptic function and  $\wp(z)$  be the Weierstrassian elliptic function discussed in §4, and let both these functions have the same two periods  $2\omega_1$  and  $2\omega_2$ .

Now, take  $f(z)$  to be the following function:

$$f(z) = \frac{1}{2}[f(z) + f(-z)] + \frac{1}{2}[\{f(z) - f(-z)\}\{\wp'(z)\}^{-1}]\wp'(z). \quad (6.1)$$

We can then say that  $f(z)$  is even, and so the functions

$$f(z) + f(-z) \quad \text{and} \quad \{f(z) - f(-z)\}\{\wp'(z)\}^{-1} \quad (6.2)$$

are both even functions.

Also, it is clear that if  $f(z)$  is an elliptic function, these two functions are elliptic functions too.

Theorem 15 will be effected if we can express any even elliptic function  $\phi(z)$ , say, in terms of  $\wp(z)$  [4].

If we let  $a$  be a zero of  $\phi(z)$  in any cell, then the point in the cell congruent to  $-a$  is a zero too. From this, the irreducible zeros of  $\phi(z)$  may be divided into two sets as follows:  $a_1, a_2, \dots, a_n$  and certain points congruent to  $-a_1, -a_2, \dots, -a_n$ .

Similarly, the irreducible poles of  $\phi(z)$  may be divided:  $b_1, b_2, \dots, b_n$  and certain points congruent to  $-b_1, -b_2, \dots, -b_n$ .

Consider the function:

$$\frac{1}{\phi(z)} \prod_{r=1}^n \left\{ \frac{\wp(z) - \wp(a_r)}{\wp(z) - \wp(b_r)} \right\}. \quad (6.3)$$

Function (6.3) is an elliptic function of  $z$ .

Note that we know the poles and zeros of  $\wp(z)$  and the product of function (6.3) holds the same ones (with same order of multiplicity) as the function  $\phi(z)$ . Since the latter one is dividing the product, the function has no poles and, by Proposition 9, is actually a constant.

Thus,

$$\phi(z) = A \prod_{r=1}^n \left\{ \frac{\wp(z) - \wp(a_r)}{\wp(z) - \wp(b_r)} \right\}, \quad (6.4)$$

where  $A$  is the aforementioned constant.

So we have expressed  $\phi(z)$  as a rational function of  $\wp(z)$ .

Applying this same process to each of the functions (6.2) we obtain Theorem 15, as desired.

## 6.2 Application

From the proof of Theorem 15 we can deduce that if we have the poles and zeros of a function, then we can write the elliptic function using (6.4):

$$\phi(z) = A \prod_{r=1}^n \left\{ \frac{\wp(z) - \wp(a_r)}{\wp(z) - \wp(b_r)} \right\},$$

where  $a_i$  and  $b_i$  are the zeros and the poles of the function, respectively.

*Example 16.* Construct an elliptic function  $f(z)$  with zeros at  $z = 1 \pm i$  and poles at  $z = 0$  and  $z = i$ .

Let us check if

$$f(z) = \frac{[\wp(z) - \wp(1+i)][\wp(z) - \wp(1-i)]}{\wp(z) - \wp(i)} \quad (6.5)$$

would be correct.

Observe how, if  $z = 1 + i$  then the left multiplier of the numerator becomes zero, making  $f(z) = 0$ , and therefore  $z = 1 + i$  is a zero of  $f(z)$ . Similarly, if  $z = 1 - i$ , the right multiplier of the numerator becomes zero, making  $f(z) = 0$ , and so  $z = 1 - i$  is also a zero of  $f(z)$ .

To check that 0 and  $i$  are poles of  $f(z)$ , first set  $z = i$ . The denominator becomes zero, so  $z = i$  is a pole of  $f(z)$ . Similarly, consider when  $z$  tends to zero. We know that  $z = 0$  is a pole of  $\wp(z)$ , since  $\wp(z) \rightarrow \infty$  as  $z \rightarrow 0$ . When this occurs,  $\wp(1+i)$ ,  $\wp(1-i)$  and  $\wp(i)$  become negligible and we have

$$\frac{\wp^2(z)}{\wp(z)} = \wp(z),$$

which tends to infinity as  $z$  tends to zero. So  $z = 0$  is also a pole of  $f(z)$ .

Thus, (6.5) is indeed the elliptic function we were looking for.

# Chapter 7

## Addition theorems

In §6 it was shown that any elliptic function can be expressed in terms of the Weierstrass  $\wp(z)$  function and its derivative. For this reason, when we discuss theorems in relation to elliptic functions in  $\wp(z)$ , we know that they apply to all elliptic functions.

All elliptic functions have addition theorems. In this chapter we aim to study some forms of these. These theorems show that it is possible to express  $\wp(u+v)$  in terms of  $\wp(u)$ ,  $\wp(v)$  and their derivatives, for general values of  $u$  and  $v$  [10].

### 7.1 Addition theorem 1

**Theorem 17.** 
$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(u+v) & -\wp'(u+v) & 1 \end{vmatrix} = 0 \text{ [4].}$$

*Proof.* Consider the following equations:

$$\wp'(u) = A\wp(u) + B \quad \text{and} \quad \wp'(v) = A\wp(v) + B. \quad (7.1)$$

These equations determine the constants  $A$  and  $B$  in terms of  $u$  and  $v$  unless  $u \equiv \pm v \pmod{2\omega_1, 2\omega_2}$  (this is, unless  $\wp(u) = \wp(v)$ ).

Now consider the function of  $\xi$ :

$$\wp'(\xi) = A\wp(\xi) + B. \quad (7.2)$$

The function  $\wp'(\xi)$  has a triple pole at  $\xi = 0$  and, therefore, by Proposition 11, it has exactly three irreducible zeros. The sum of these zeros forms a period and, given that  $\xi = u$  and  $\xi = v$  are two zeros, the third zero, say  $\xi = w$ , is such that  $w = -u - v \pmod{2\omega_1, 2\omega_2}$ .

This implies that  $w$  is a zero of the function (7.2), which then gives the following:

$$\wp'(w) = A\wp(w) + B. \quad (7.3)$$

Eliminating  $A$  and  $B$  throughout, the discriminant obtained is:

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(-w) & -\wp'(-w) & 1 \end{vmatrix} = 0,$$

which implies

$$\wp(u)[\wp'(v) + \wp'(-w)] - \wp'(u)[\wp(v) - \wp(-w)] + [-\wp(v)\wp'(-w) - \wp'(v)\wp(-w)] = 0.$$

Which, using the oddity of  $\wp'(z)$  and evenness of  $\wp(z)$  (this is,  $\wp'(z) = -\wp'(-z)$  and  $\wp(z) = \wp(-z)$ ), can be written as

$$\wp(u)\wp'(v) - \wp(u)\wp'(w) - \wp'(u)\wp(v) + \wp'(u)\wp(w) + \wp(v)\wp'(w) - \wp'(v)\wp(w) = 0.$$

Since this result expresses  $\wp(u+v)$  algebraically in terms of  $\wp(u)$  and  $\wp(v)$ , the Addition Theorem 17 holds, where  $u+v = -w$ .  $\square$

## 7.2 Addition theorem 2

**Theorem 18.**  $\wp(u+v) = \frac{1}{4} \left[ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2 - \wp(u) - \wp(v)$  [4].

*Proof.* Consider once more the function (7.2). When  $\xi$  is congruent to any of the zeros  $\xi = u$ ,  $\xi = v$  or  $\xi = -u - v = w$ , squaring the function and rearranging it gives:

$$\wp'^2(\xi) - [A\wp(\xi) + B]^2 = 0.$$

If  $\wp(\xi)$  is equal to any of  $\wp(u)$ ,  $\wp(v)$  or  $\wp(-w)$ , then

$$4\wp^3(\xi) - A^2\wp^2(\xi) - (2AB + g_2)\wp(\xi) + (B^2 + g_3) = 0, \quad (7.4)$$

where  $g_2$  and  $g_3$  are as given in §4.2.

$\wp(u)$ ,  $\wp(v)$  and  $\wp(-w)$  are all unequal for general values of  $u$  and  $v$ , and so they are all roots of (7.4).

Consequently, using the formula for the sum of the roots of a cubic equation, we obtain

$$\wp(u) + \wp(v) + \wp(-w) = \frac{1}{4}A^2. \quad (7.5)$$

By means of rearranging (7.5),

$$\wp(-w) = \frac{1}{4} \left[ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2 - \wp(u) - \wp(v),$$

which, if you substitute  $w$  for  $-u - v$ , becomes the Addition Theorem 18:

$$\wp(u + v) = \frac{1}{4} \left[ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2 - \wp(u) - \wp(v).$$

This result expresses explicitly  $\wp(u + v)$  in terms of  $\wp(u)$ ,  $\wp(v)$  and their derivatives, as desired.  $\square$

### 7.3 Other addition theorems

Many algebraic addition theorems for elliptic functions can be found. Here are two other such examples.

*Example 19.*  $\wp'(u + v) = \frac{\wp(u+v)[\wp'(u) - \wp'(v)] + \wp(u)\wp'(v) - \wp'(u)\wp(v)}{\wp(v) - \wp(u)}$  [7].

*Example 20.*  $\wp(u + v) - \wp(u - v) = \frac{-\wp'(u)\wp'(v)}{[\wp(u) - \wp(v)]^2}$  [4].

To prove these, we can account for poles and zeros on the left-hand side of the identities and check these are the same on the right-hand side (or vice versa), and confirm that Theorem 5 holds by studying the behaviour of both sides of the identity at a chosen limiting point.

For instance, Example 20 may be proved in the following way.

*Proof.* Let us assume  $v$  is a constant.

Consider first the left-hand side of the identity,

$$\wp(u+v) - \wp(u-v). \quad (7.6)$$

$z$  is a zero of (7.6) if (7.6) equals zero when  $u = z$ .

- $u = 0 \Rightarrow \wp(0+v) - \wp(0-v) = \wp(v) - \wp(-v) = \wp(v) - \wp(v) = 0$ ;
- $u = \omega_1 \Rightarrow \wp(\omega_1+y) - \wp(\omega_1-y) = \wp(\omega_1+y) - \wp(-\omega_1+y) = \wp(\omega_1+y) - \wp(-\omega_1+y+2\omega_1) = \wp(\omega_1+y) - \wp(\omega_1+y) = 0$ ;

and similarly for  $u = \omega_2$  and  $u = \omega_3$ . So  $0, \omega_1, \omega_2$  and  $\omega_3$  are all zeros of (7.6).

Let us check these are the zeros of the right-hand side of the identity,

$$\frac{-\wp'(u)\wp'(v)}{[\wp(u) - \wp(v)]^2}. \quad (7.7)$$

- $u = 0 \Rightarrow \frac{-\wp'(0)\wp'(v)}{[\wp(0) - \wp(v)]^2} = 0$  since  $\wp'(0) = 0$ ;
- $u = \omega_1 \Rightarrow \frac{-\wp'(\omega_1)\wp'(v)}{[\wp(\omega_1) - \wp(v)]^2} = 0$  since  $\wp'(\omega_1) = 0$ ;

and similarly for  $u = \omega_2$  and  $u = \omega_3$ . So the zeros are the same for both sides of the identity.

Now let us look for poles. These will be the points that will make both sides of the identity tend to infinity. In (7.6) we deal with  $\wp(z)$ . From equation (4.1) we know that it tends to infinity when  $z = 0$ . So (7.6) tends to infinity when  $u = \pm v$ . These are double poles, so the number of poles is equal to the number of zeros, and we are done looking for poles.

Let us confirm  $u = \pm v$  are also poles of (7.7). The function will tend to infinity when  $\wp(u) - \wp(v)$  is equal to zero. This occurs when  $u = \pm v$ , and the square in the denominator makes these double poles.

Finally, let us consider the behaviour of the identity when  $u$  tends to zero. When  $u \rightarrow 0$ ,  $\wp(u) \sim z^{-2}$  and  $\wp'(u) \sim -2z^{-3}$ .

$$\wp(u+v) \simeq \wp(v) + \wp'(v)u + O(u^2)$$

and

$$\wp(u - v) = \wp(v - u) \simeq \wp(v) - \wp'(v)u + O(u^2),$$

so  $\wp(u + v) - \wp(u - v) \simeq 2\wp'(v)u$ . So (7.6) behaves like  $2u$ .

Now, (7.7) behaves like  $\frac{2z^{-3}}{[z^{-2}]^2} = 2z$ , so both sides of the identity behave in the same manner.

Thus it is proved that the addition theorem of Example 20 holds. □

# Chapter 8

## The Weierstrass theorem

Definition 3 told us what a meromorphic function is. Now we want to relate these functions to algebraic addition theorems.

So, a meromorphic function  $\phi(z)$  has an algebraic addition theorem if there exists an identity

$$F(\phi(z_1 + z_2), \phi(z_1), \phi(z_2)) = 0, \quad (8.1)$$

where  $F$  is an entire rational function of its arguments [5]. From equation (8.1) we understand that  $\phi(z_1 + z_2)$  can be expressed algebraically in terms of  $\phi(z_1)$  and  $\phi(z_2)$ .

In §7 we said that all elliptic functions possess algebraic addition theorems. These also hold for degeneracies of elliptic functions (this is, their limiting cases), such as any rational function of  $z$  and also any rational function of  $e^{\frac{i\pi z}{\omega}}$ .

The following theorem (given by Weierstrass) states that, in fact, the above two examples are the only meromorphic functions that have an algebraic addition theorem.

**Theorem 21.** *Any meromorphic function  $\phi(z)$  possessing an algebraic addition theorem is either an elliptic function or is of the form  $R(z)$  or  $R(e^{\lambda z})$ , where  $R$  is a rational function [11].*

Note that in Theorem 21,  $\lambda = \frac{i\pi}{\omega}$ .

We will continue to prove Weierstrass' Theorem with the aid of W. S. Osgood's proofs in [5] and [11], after stating a necessary theorem.

## 8.1 Picard's Theorem

The following theorem is known as the Big Picard Theorem.

**Theorem 22.** *Any analytic function  $\phi(z)$  assumes in an arbitrary neighbourhood of an essentially singular point any finite value except, perhaps, one value [11].*

So if an entire function omits two (or more) values, then it is a constant.

## 8.2 The proof

The following is the proof of the Weierstrass Theorem (Theorem 21).

*Proof.* Suppose that the meromorphic function  $\phi(z)$  has an algebraic addition theorem as (8.1). Assume that  $F(x_1, x_2, x_3)$  is a polynomial in  $x_1, x_2, x_3$  and, as a polynomial in  $x_1$  (for us,  $x_1 = \phi(z_1 + z_2)$ ), the degree is  $m$ . We have to show that  $\phi(z)$  is either rational, elliptic or simply periodic. Suppose it is not rational and thus,  $z = \infty$  is an essential singularity of  $\phi(z)$ .

We choose a  $C$  such that  $\phi(z) = C$  has many solutions (and  $C$  is not omitted by  $\phi(z)$ ). Using the Big Picard Theorem (Theorem 22), we know there are  $m + 1$  different solutions to this equation, and we shall denote them by  $a_0, a_1, a_2, \dots, a_m$ . Also, choose a  $\xi$  such that  $\xi, a_0 + \xi, a_1 + \xi, a_2 + \xi, \dots, a_m + \xi$  are all regular points.

Now consider the following equation:

$$F(x, \phi(z), C) = 0, \tag{8.2}$$

where  $z$  is one of the points  $a_k + \xi$  ( $k = 0, 1, 2, \dots, m$ ). Setting  $z_1 = z$  and  $z_2 = a_k$  in equation (8.1) we get:

$$F(\phi(z + a_k), \phi(z), \phi(a_k)) = 0. \tag{8.3}$$

If we write  $C = \phi(a_k)$  and  $x = \phi(z + a_k)$ , we obtain equation (8.2). So (8.2) has roots for  $x = \phi(z + a_k)$  ( $k = 0, 1, 2, \dots, m$ ). Observe that since  $k = 0, 1, 2, \dots, m$ , there are  $m + 1$  such solutions.

Since  $F(x, \phi(z), C)$  is a polynomial in  $x$  of degree  $m$ , the roots of  $x = \phi(z + a_k)$  ( $k = 0, 1, 2, \dots, m$ ) cannot all be distinct, so

$$\phi(z + a_r) = \phi(z + a_s) \tag{8.4}$$

for each  $z$  in some neighbourhood of  $\xi$  and  $r = 0, 1, 2, \dots, m$  and  $s = 0, 1, 2, \dots, m$  but  $r \neq s$ . From this, we can state  $\phi(z)$  to have period  $a_r - a_s$ , which means that  $\phi(z)$  is periodic.

If  $\phi(z)$  is doubly periodic, from the definition of an elliptic function (Definition 4), the theorem is proved. So it remains to be checked what happens if  $\phi(z)$  is simply periodic.

For simplicity, assume that the period of  $\phi(z)$  is equal to  $2\pi$ . We want to show that  $\phi(z) = R(\omega)$ , where  $\omega = e^{iz}$  and  $R$  represents a rational function. The map  $z \mapsto \omega = e^{iz}$  sends the strip  $0 < \Re(z) < 2\pi$  into the  $\omega$ -plane, with the cut from 0 to  $+\infty$ . We can see that  $\phi(z) = \psi(\omega)$  is meromorphic in the whole  $\omega$ -plane and has punctured points at  $z = 0$  and  $z = \infty$ . If neither of these points are essential singularities, then the function  $\psi(z)$  is rational. But if we assume that at least one of these two points is an essential singularity, then we can apply Theorem 22 once again to find  $m+1$  distinct points  $b_k$  ( $k = 0, 1, 2, \dots, m$ ) at which  $\psi(\omega) = C$ , as before. The inverse images of these  $b_0, b_1, b_2, \dots, b_m$  lie in the strip  $0 \leq \Re(z) < 2\pi$ , showing that  $\phi(z)$  as presented in equation (8.2) has a period whose real part is  $< 2\pi$ , which is a contradiction to our initial assumption, that is that the function  $\phi(z)$  has a single period of  $2\pi$ .

Therefore, the Weierstrass Theorem is proved. □

# Chapter 9

## Some consequences of $\omega_2 \rightarrow \infty$

As a final note for this dissertation, we will study the limits, when  $\omega_2$  tends to infinity, of  $g_2$ ,  $g_3$  and the Weierstrass  $\wp(z)$  function.

### 9.1 $g_{2_\infty}$ and $g_{3_\infty}$

Recall from §4.2 that

$$g_2 = 60 \sum'_{m,n} \frac{1}{(2m\omega_1 + 2n\omega_2)^4}$$

and

$$g_3 = 140 \sum'_{m,n} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}.$$

Let  $g_{2_\infty}$  and  $g_{3_\infty}$  be the limits of  $g_2$  and  $g_3$  when  $\omega_2$  tends to infinity, respectively.

$$\begin{aligned}
g_{2_\infty} &= \lim_{\omega_2 \rightarrow \infty} g_2 \\
&= 60 \sum_{m \neq 0} \frac{1}{(2m\omega_1)^4} \\
&= 120 \sum_{m=1}^{\infty} \frac{1}{(2m\omega_1)^4} \\
&= \frac{15}{2\omega_1^4} \sum_{m=1}^{\infty} \frac{1}{m^4} \\
&= \frac{15}{2\omega_1^4} \frac{\pi^4}{90} \\
&= \frac{\pi^4}{12\omega_1^4}.
\end{aligned}$$

$$\begin{aligned}
g_{3_\infty} &= \lim_{\omega_2 \rightarrow \infty} g_3 \\
&= 140 \sum_{m \neq 0} \frac{1}{(2m\omega_1)^6} \\
&= 280 \sum_{m=1}^{\infty} \frac{1}{(2m\omega_1)^6} \\
&= \frac{35}{8\omega_1^6} \sum_{m=1}^{\infty} \frac{1}{m^6} \\
&= \frac{35}{8\omega_1^6} \frac{\pi^6}{945} \\
&= \frac{\pi^6}{216\omega_1^6}.
\end{aligned}$$

As expected, from the definitions of  $g_2$  and  $g_3$ , letting  $\omega_2$  tend to infinity expresses  $g_{2_\infty}$  and  $g_{3_\infty}$  only in terms of the remaining period,  $2\omega_1$ .

They differ by a factor of  $\frac{\pi^2}{18\omega_1^2}$ . Both  $g_{2_\infty}$  and  $g_{3_\infty}$  will tend to zero as  $\omega_1$  tends to infinity, but  $g_{3_\infty}$  will tend to this limit faster.

## 9.2 $\wp_\infty(z)$

Let  $\wp_\infty(z)$  be the limit of the Weierstrass  $\wp(z)$  function when  $\omega_2$  tends to infinity.

$$\begin{aligned}\wp_\infty(z) &= \lim_{\omega_2 \rightarrow \infty} \wp(z) \\ &= \frac{1}{z^2} + \sum_{m \neq 0} \left[ \frac{1}{(z - 2m\omega_1)^2} - \frac{1}{(2m\omega_1)^2} \right] \\ &= \sum_{m \in \mathbb{Z}} \frac{1}{(z - 2m\omega_1)^2} - \frac{1}{2\omega_1^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \\ &= -\frac{\pi^2}{4\omega_1^2(-1 + \cos(\frac{\pi z}{2\omega_1}))^2} - \frac{\pi^2}{6}.\end{aligned}$$

It is interesting to see that making one of the periods of the Weierstrass  $\wp(z)$  function tend to infinity, the expression obtained is one involving the trigonometric function of cosine with respect to the inverse of the period that is left.

# Chapter 10

## Conclusion

We have then discussed the basic properties of elliptic functions and studied some interesting theorems and propositions that surround them.

We have learnt about the simplest non-trivial elliptic function, the Weierstrass  $\wp(z)$  function, and how all elliptic functions can be expressed by means of this particular one. We have also fully proved the Weierstrass Theorem on meromorphic functions having algebraic addition theorems.

It has been a very analytic paper. For the study of some "real life" applications, I recommend the book *Elliptic functions and applications* by D. F. Lawden [8].

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